Ramanujan’s eta-products and binary quadratic forms

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Definitions

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We will use the standard notation of

$$ (a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n), \quad (1.1) $$

$$ E(q) := (q; q)_{\infty}, \quad (1.2) $$
Define $a_k(n)$ by

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Ramanujan found the Dirichlet series $\sum_{n>0} \frac{a_k(n)}{n^s}$ has an Euler product when $k = 1, 2, 3, 4, 6, 8, 12$ [R1], [R2]. He explicitly gave the Euler product for $k = 1, 2, 3$ and determined the Fourier coefficients of the product.
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We now develop the theory of binary quadratic forms and find the connection between weight 1 eta-products and representations by forms.
For $a, b, c \in \mathbb{Z}$ we denote the binary quadratic form $f(x, y) := ax^2 + bxy + cy^2$ as $(a, b, c)$. 
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The \textit{discriminant} of $(a, b, c)$ is defined to be $d := b^2 - 4ac$, and we note that $d \equiv 0, 1 \pmod{4}$. 
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It is easy to show $d < 0 < a$ implies $f(x, y) \geq 0$ for all $x, y \in \mathbb{Z}$. Such forms are called positive definite. This presentation only considers positive definite forms.
We say two binary quadratic forms \( f(x, y), g(x, y) \) are equivalent if there exists a matrix

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\]

in \( SL(2\mathbb{Z}) \) such that

\[ f(\alpha x + \beta y, \gamma x + \delta y) = g(x, y), \]

and we say \( f \sim g \).
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The answer is any positive definite form \(f\) has exactly 2 automorphs, except when \(f = (1, 0, 1)\) or \(f = (1, 1, 1)\).
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We define the number of automorphs associated with the discriminant \(d\) as \(w(d)\) and we find

\[
w(d) = \begin{cases} 
6 & \text{if } d = -3 \\
4 & \text{if } d = -4 \\
2 & \text{if } d < -4.
\end{cases}
\]
We define $H(d)$ to be the set of primitive binary quadratic forms of discriminant $d$ modulo the equivalence relation $\sim$. 
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We now motivate the composition (multiplication) of two binary quadratic forms.
Brahmagupta’s identity

\[(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1 x_2 + y_1 y_2)^2 + (x_1 y_2 - y_1 x_2)^2\]

was first found in Diophantus’ *Arithmetica* (*III, 19*). This identity shows the group law for the form \((1, 0, 1) \in H(−4)\).
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was first found in Diophantus’ *Arithmetica* (III, 19). This identity shows the group law for the form $(1, 0, 1) \in H(-4)$.

The above identity was rediscovered by the Indian mathematician Brahmagupta, and Brahmagupta showed more:

$$(x_1^2 + ny_1^2)(x_2^2 + ny_2^2) = (x_1x_2 + ny_1y_2)^2 + n(x_1y_2 - y_1x_2)^2.$$
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Similarly we have

\[(x_1^2 + x_1y_1 + y_1^2) \cdot (x_2^2 + x_2y_2 + y_2^2)\]

\[= X^2 + XY + Y^2,\]

where

\[X = x_1x_2 - y_1y_2, \quad Y = x_1y_2 + y_1x_2 + y_1y_2.\]
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$$(n_1, B, \frac{B^2 - d}{4n_1}) \cdot (n_2, B, \frac{B^2 - d}{4n_2}) = (n_1 n_2, B, \frac{B^2 - d}{4n_1 n_2}).$$
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Explicitly

$$(n_1x_1^2 + Bx_1y_1 + \frac{B^2 - d}{4n_1}y_1^2) \cdot (n_2x_2^2 + Bx_2y_2 + \frac{B^2 - d}{4n_2}y_2^2) = n_1n_2X^2 + BXY + \frac{B^2 - d}{4n_1n_2}Y^2$$

where

$$X = x_1x_2 - \frac{B^2 - d}{4n_1n_2}y_1y_2, \quad Y = n_1x_1y_2 + n_2y_1x_2 + By_1y_2.$$
A generalization which shows the multiplication of two forms of discriminant $d$ is given by:

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$$(n_1x_1^2 + Bx_1y_1 + \frac{B^2 - d}{4n_1}y_1^2) \cdot (n_2x_2^2 + Bx_2y_2 + \frac{B^2 - d}{4n_2}y_2^2) = n_1n_2X^2 + BXY + \frac{B^2 - d}{4n_1n_2}Y^2$$

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We see that the inverse of $(a, b, c)$ is $(c, b, a) \sim (a, -b, c)$.  
Gen. comp.
For $n \in \mathbb{N}$ we say $(a, b, c)$ represents $n$ when there exists integers $x, y$ such that $n = ax^2 + bxy + cy^2$. 
For $n \in \mathbb{N}$ we say $(a, b, c)$ represents $n$ when there exists integers $x, y$ such that $n = ax^2 + bxy + cy^2$.

We say $(a, b, c)$ properly represents $n$ when there exists coprime integers $x, y$ such that $n = ax^2 + bxy + cy^2$. 
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We define $R((a, b, c), n)$ to be the number of representations of $n$ by $(a, b, c)$.

Similarly, $R'(a, b, c), n)$ is the number of proper representations of $n$ by $(a, b, c)$. 
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Similarly, $R'((a, b, c), n)$ is the number of proper representations of $n$ by $(a, b, c)$.

Also

$$ B(a, b, c, q) := \sum_{x,y} q^{ax^2 + bxy + cy^2} = \sum_{n \geq 0} R((a, b, c), n)q^n. $$
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The functions $R$ and $R'$ play a large role in the theory of binary quadratic forms, and we now consider a few of their most basic properties.
Theorem
Let $d$ be a discriminant with conductor $f$. Let $p$ be a prime and $K \in H(d)$.
We have:

1. $p$ is represented by some class in $H(d)$ iff $(d, p) = 0$, and $p \nmid f$.
2. Suppose $p | d$ and $p \nmid f$. Then $p$ is represented by exactly one class $A \in H(d)$ and $A = A - 1$. Moreover, $R(A, p) = R'(A, p) = w(d)$.
3. Suppose $(d, p) = 1$. Then $p$ is represented by some class $A \in H(d)$, and $R(K, p) = R'(K, p) = \begin{cases} 0 & \text{if } K \neq A \\ A - 1 & \text{if } A \neq A - 1 \\ 2w(d) & \text{if } K = A = A - 1 \end{cases}$.
Theorem

Let $d$ be a discriminant with conductor $f$. Let $p$ be a prime and $K \in H(d)$.

We have:

1. $p$ is represented by some class in $H(d)$ iff $\left(\frac{d}{p}\right) = 0, 1$ and $p \nmid f$. 

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[D., S.W.]
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[D.], [S.W.]
Theorem

Let \((\alpha, \gamma)\) be a proper representation of \(n > 0\) by \((a, b, c)\) of discriminant \(d\). Then integers \(\beta, \delta, m\) can be determined in one and only one way to satisfy:

\[ m \leq m < 2n, \quad m^2 \equiv d \pmod{4n}, \quad (\alpha \beta \gamma \delta) \in SL(2, \mathbb{Z}) \text{ takes } (a, b, c) \text{ to } (n, m, m^2 - d/4n).\]
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Let $f$ be a positive definite binary quadratic form of discriminant $d$. For $n \in \mathbb{N}$ let $H_f(n)$ be the number of integers $m$ satisfying

- $0 \leq m < 2n$,
- $m^2 \equiv d \pmod{4n}$,
- $(n, m, \frac{m^2-d}{4n}) \sim f$.

Theorem
With $f$ as above, we have

$$R'(f, n) = w(d)H_f(n),$$

and

$$R(f, n) = \sum_{k^2|n} R'(f, n/k^2).$$
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Theorem

Let $d$ be a discriminant. If $n_1, n_2$ are pairwise prime positive integers and $K \in H(d)$, then

$$R(K, n_1 \cdot n_2) = \frac{1}{w(d)} \sum_{K_1 \cdot K_2 = K} R(K_1, n_1) \cdot R(K_2, n_2)$$

where the summation is taken over all $K_1, K_2 \in H(d)$ such that $K_1 \cdot K_2 = K$ [S.W.].
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where the summation is taken over all $K_1, K_2 \in H(d)$ such that $K_1 \cdot K_2 = K$ [S.W.].

The proof hinges on the lemma

$$H_K(n_1 \cdot n_2) = \sum_{K_1 \cdot K_2 = K} H_{K_1}(n_1) \cdot H_{K_2}(n_2)$$

which Habib and Williams give in [H.W.]. The proof is elementary yet somewhat technical.
Proof of Theorem:

\[
\frac{R(K, n_1 \cdot n_2)}{w(d)} = \sum_{m^2 | n_1 \cdot n_2} H_K \left( \frac{n_1 \cdot n_2}{m^2} \right)
\]
Proof of Theorem:

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\[
= \sum_{m_1^2 | n_1} \sum_{m_2^2 | n_2} H_K \left( \frac{n_1}{m_1^2} \cdot \frac{n_2}{m_2^2} \right)
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Proof of Theorem:

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\frac{R(K, n_1 \cdot n_2)}{w(d')} = \sum_{m^2 | n_1 \cdot n_2} H_K \left( \frac{n_1 \cdot n_2}{m^2} \right)
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Proof of Theorem:

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\[
= \sum_{m_1^2 | n_1} \sum_{m_2^2 | n_2} H_K \left( \frac{n_1}{m_1^2} \cdot \frac{n_2}{m_2^2} \right)
\]

\[
= \sum_{m_1^2 | n_1} \sum_{m_2^2 | n_2} \sum_{K_1 \cdot K_2 = K} H_{K_1} \left( \frac{n_1}{m_1^2} \right) H_{K_2} \left( \frac{n_2}{m_2^2} \right)
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Definition

Let $d$ be a discriminant and $n \in \mathbb{N}$. Let

$$H(d) = \{A_1^{k_1} \cdots A_r^{k_r} | 0 \leq k_1 < h_1, \ldots, 0 \leq k_r < h_r\}$$

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For \( K = A_1^{k_1} \cdots A_r^{k_r} \in H(d) \) and \( M = A_1^{m_1} \cdots A_r^{m_r} \in H(d) \),
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\[ \langle K, M \rangle = \frac{k_1 m_1}{h_1} + \cdots + \frac{k_r m_r}{h_r}. \]
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We define

$$F(M, n) = \frac{1}{w(d)} \sum_{K \in H(d)} \cos 2\pi \langle K, M \rangle \cdot R(K, n).$$
Theorem

$F(M, n)$ is a multiplicative function of $n$. 
Ramanujan's eta-products and binary quadratic forms

Frank Patane

**Theorem**

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**Proof:**

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F(M, n) = \frac{1}{w(d)} \sum_{K \in H(d)} \cos 2\pi \langle K, M \rangle \cdot R(K, n)
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Hecke Operators

In many cases, $F(M, n)$ has the additional property of being an eigenform for all Hecke operators.
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Let $(a, b, c)$ have the associated theta series $\sum_{n>0} h(n)q^n$. In [He.], Hecke defines the operator $T_p$ as

$$T_p \left( \sum_{n=0}^{\infty} h(n)q^n \right) = \sum_{n=0}^{\infty} (h(pn) + \left( \frac{-d}{p} \right) h(n/p))q^n. \quad (4.1)$$

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If $T_p(s) = s \cdot \lambda_p$ for some constant $\lambda_p$, then we call $s$ an eigenform of $T_p$ with eigenvalue $\lambda_p$. 
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Furthermore \( h \) satisfies

\[
h(p^{k+1}) = h(p)h(p^k) - \left( \frac{-d}{p} \right) h(p^{k-1}).
\]

(4.3)
We now give an example showing multiplicativity does not imply being an eigenform for all $T_p$. 

Let us take $d = -224 = 2^5 \cdot 7$. 

Principal Genus $(1, 0, 56), (8, 8, 9)$ 

Second Genus $(4, 4, 15), (7, 0, 8)$ 

Third Genus $(3, 2, 19), (3, -2, 19)$ 

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<tbody>
<tr>
<td><strong>CL($-224$)</strong></td>
<td>$\cong \mathbb{Z}_4 \times \mathbb{Z}_2$</td>
<td></td>
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For any form $A$ from the third or fourth genus above,

$$F(A, n) = \frac{1}{2} \left[ R((1, 0, 56), n) - R((8, 8, 9), n) + R((4, 4, 15), n) - R((7, 0, 8), n) \right]$$
\( F(A, n) \) is multiplicative yet \( T_2 \) acting on \( \sum_{n>0} F(A, n)q^n \) gives

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\sum_{n>0} [R((1, 0, 14), n) - R((2, 0, 7), n)]q^{2n}.
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The above failure of $F(A, n)$ to be an eigenform under $T_2$ is not caused simply because $p = 2$.

Indeed, for any finite list of primes, $\{p_1, \ldots, p_k\}$, we can find a discriminant such that at least one multiplicative combination $F(M, n)$ will fail to be an eigenform under $T_{p_i}$ for all $1 \leq i \leq k$. 
When \( F(M, n) \) enjoys the property of being an eigenform for all Hecke operators, computing \( F(M, p) \) for all primes \( p \) suffices to compute \( F(M, n) \).
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Before we revisit the eta-products discussed by Ramanujan, we give a theorem which connects binary quadratic forms to eta-products.
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Before we revisit the eta-products discussed by Ramanujan, we give a theorem which connects binary quadratic forms to eta-products.

**Theorem**

If $m, s$ are positive integers with $24s - m > 0$, then

$$
\frac{B(6m, m, s, q) - B(6m, 5m, s + m, q)}{2} = q^s E(q^m)E(q^{24s-m}).
$$

(5.1)
In the beginning of the talk we defined $a_m(n)$ by

$$\sum_{n=1}^{\infty} a_m(n)q^n := qE(q^m)E(q^{24-m}).$$

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Setting $s = 1$ in the previous Theorem we arrive at

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\frac{B(6m, m, 1, q) - B(6m, 5m, 1 + m, q)}{2} = qE(q^m)E(q^{24-m}).
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for $m = 1, \ldots, 12$. 

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Ramanujan saw that for $m = 1, 2, 3, 4, 6, 8, 12$, the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_m(n)}{n^s}$ has an Euler product.
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Ramanujan saw that for $m = 1, 2, 3, 4, 6, 8, 12$, the Dirichlet series $\sum_{n=1}^\infty \frac{a_m(n)}{n^s}$ has an Euler product.

In other words, $a_m(n)$ is multiplicative for $m = 1, 2, 3, 4, 6, 8, 12$. 
For $s = 1$ and $d = -m(24 - m)$ we have

| $m$ | $d$ | $|H(m)|$ |
|-----|-----|---------|
| 1   | -23 | 3       |
| 2   | -44 | 3       |
| 3   | -63 | 4       |
| 4   | -80 | 4       |
| 5   | -95 | 8       |
| 6   | -108| 3       |
| 7   | -119| 10      |
| 8   | -128| 4       |
| 9   | -135| 6       |
| 10  | -140| 6       |
| 11  | -143| 10      |
| 12  | -144| 4       |
When $m = 1, 2, 3, 4, 6, 8, 12$ we get that $H(m)$ is cyclic of order 3 or 4 and,
When \( m = 1, 2, 3, 4, 6, 8, 12 \) we get that \( H(m) \) is cyclic of order 3 or 4 and,

\[
F(A, n) = \frac{R((6m, m, 1, n)) - R((6m, 5m, 1 + m, n))}{2} \quad (5.4)
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where \( A \) is a generator of \( H(m) \).
When \( m = 1, 2, 3, 4, 6, 8, 12 \) we get that \( H(m) \) is cyclic of order 3 or 4 and,

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where \( A \) is a generator of \( H(m) \).

The \( F(A, n) \) in (5.4) are eigenforms for all Hecke operators and we can easily deduce their values and write an Euler product for the corresponding Dirichlet series.
Let us consider $d = -135$. Theorem (8) with $s = 1$ and $m = 9$ gives

$$\sum_{n=1}^{\infty} a(n)q^n := \sum_{n=1}^{\infty} \frac{R((1, 1, 34), n) - R((4, 3, 9), n)}{2} q^n$$  \hspace{1cm} (6.1)

$$= qE(q^9)E(q^{15}),$$  \hspace{1cm} (6.2)
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We note that the above $a(n)$ is not multiplicative.
Discriminant $-135$ is isomorphic to $\mathbb{Z}_6$:

\[
\begin{array}{|c|c|c|c|}
  \hline
  & \text{CL}(-135) \cong C_6 & (\frac{p}{5}) & (\frac{p}{3}) \\
  \hline
  \text{Principal Genus} & (1, 1, 34), (4, 3, 9), (4, -3, 9) & +1 & +1 \\
  \hline
  \text{Second Genus} & (5, 5, 8), (2, 1, 17), (2, -1, 17) & -1 & -1 \\
  \hline
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\]
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We find that

$$F((2, 1, 17), n) = \frac{R((1, 1, 34), n) - R((4, 3, 9), n)}{2} + \frac{R((2, 1, 17), n) - R((5, 5, 8), n)}{2}.$$  \hspace{1cm} (6.3)

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For a class group isomorphic to $\mathbb{Z}_6$, $F(A, n)$ is no longer just a simple difference of two theta series.
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We need to also employ the product

$$\sum_{n=1}^{\infty} b(n)q^n := \sum_{n=1}^{\infty} \frac{R((2, 1, 17), n) - R((5, 5, 8), n)}{2} q^n \quad (6.5)$$

$$= q^2 E(q^3) E(q^{45}). \quad (6.6)$$
For a class group isomorphic to $\mathbb{Z}_6$, $F(A, n)$ is no longer just a simple difference of two theta series.

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$$
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$$

We find that both $qE(q^9)E(q^{15}) \pm q^2 E(q^3)E(q^{45})$ are eigenforms for all Hecke operators. Thus we can find the coefficients of $qE(q^9)E(q^{15})$ and $q^2 E(q^3)E(q^{45})$. Explicit.
Thank you!!


Reduction of forms

We would like to have a canonical representative for a given equivalence class of binary quadratic forms.
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The positive definite form \((a, b, c)\) is said to be reduced if

\[-|a| < b \leq |a| < |c|,
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or

\[0 \leq b \leq |a| = |c|.
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The positive definite form \((a, b, c)\) is said to be *reduced* if

\[ -|a| < b \leq |a| < |c|, \]

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\[ 0 \leq b \leq |a| = |c|. \]

Each equivalence class of positive definite binary quadratic forms has a unique reduced form [B.], [D.].
General Composition

Let \((a_1, b_1, c_1)\) and \((a_2, b_2, c_2)\) be two binary quadratic forms of discriminant \(d\). Take \(t = \gcd(a_1, a_2, (b_1 + b_2)/2)\), and let \(u, v, w\) be integers such that

\[
a_1 u + a_2 v + \frac{b_1 + b_2}{2} w = t.
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General Composition

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a_1 u + a_2 v + \frac{b_1 + b_2}{2} w = t.
\]

Then \((a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = (a_3, b_3, c_3)\), where

\[
a_3 = \frac{a_1 a_2}{t^2},
\]

\[
b_3 = b_2 + \frac{a_2(b_1 - b_2)v - 2a_2c_2w}{t},
\]

\[
c_3 = \frac{b_3^2 - d}{4a_3},
\]

[B.]. Back to presentation.
The eigenvalues for $qE(q^9)E(q^{15}) \pm q^2 E(q^3)E(q^{45})$ will depend on which form if any represents the prime $p$. 
The eigenvalues for $qE(q^9)E(q^{15}) \pm q^2E(q^3)E(q^{45})$ will depend on which form if any represents the prime $p$.

We can examine the factorization of the Weber class polynomial $W_{135}(x) = x^6 - 3x^5 - 3x^4 + 2x^3 + 30x^2 + 33x - 1$ modulo $p$ to determine which form represents $p$: 
The eigenvalues for \( qE(q^9)E(q^{15}) \pm q^2 E(q^3)E(q^{45}) \) will depend on which form if any represents the prime \( p \).

We can examine the factorization of the Weber class polynomial \( W_{135}(x) = x^6 - 3x^5 - 3x^4 + 2x^3 + 30x^2 + 33x - 1 \) modulo \( p \) to determine which form represents \( p \):

For a prime \( p \) with \( \left( \frac{-135}{p} \right) = 1 \) we have

1. \( p \) is represented by the form \((1, 1, 34)\) if and only if \( W_{135}(x) \) splits completely modulo \( p \).

2. \( p \) is represented by the form \((4, 3, 9)\) if and only if \( W_{135}(x) \) factors into two irreducible cubic polynomials modulo \( p \).

3. \( p \) is represented by the form \((5, 5, 8)\) if and only if \( W_{135}(x) \) factors into three irreducible quadratic polynomials modulo \( p \).

4. \( p \) is represented by the form \((2, 1, 17)\) if and only if \( W_{135}(x) \) remains irreducible modulo \( p \).
We derive

\[ a(p^\alpha) = \begin{cases} 
0 & p = 3, \ \alpha > 0, \\
(-1)^\alpha & p = 5, \\
1 + \alpha & (1, 1, 34, p) > 0, \\
(-1)^\alpha(1 + \alpha) & p \neq 5 \ (5, 5, 8, p) > 0, \\
U(\alpha) & (4, 3, 9, p) > 0, \\
V(\alpha) & (2, 1, 17, p) > 0, \\
\frac{1 + (-1)^\alpha}{2} & \left(\frac{-135}{p}\right) = -1, \\
\end{cases} \] (8.1)

and
Ramanujan's eta-products and binary quadratic forms

Frank Patane

\[ b(p^\alpha) = \begin{cases} 
0 & p = 3, \quad \alpha > 0, \\
1 & p = 5, \\
1 + \alpha & p \neq 5, \quad (1, 1, 34, p) + (5, 5, 8, p) > 0, \\
U(\alpha) & (4, 3, 9, p) + (2, 1, 17, p) > 0, \\
\frac{1+(-1)^\alpha}{2} & \left(\frac{-135}{p}\right) = -1,
\end{cases} \]

(8.2)

where

\[ U(z) := \frac{\sin(2\pi(z+1)/3)}{\sin(2\pi/3)}, \]

(8.3)

and

\[ V(z) := \frac{\sin(\pi(z+1)/3)}{\sin(\pi/3)}. \]

(8.4)
We find

\[ \sum_{n>0} \frac{b(n)}{n^s} = \left( \frac{1}{1 - 5^{-s}} \right) \cdot \prod_r \frac{1}{(1 - r^{-s})^2} \times \prod_q \frac{1}{1 + q^{-s} + q^{-2s}} \cdot \prod_p \frac{1}{1 - p^{-2s}} \]  

(8.5) 

where

- \( r \neq 5 \) is prime with \((1, 1, 34, r) + (5, 5, 8, r) > 0\)
- \( q \) is prime with \((4, 3, 9, q) + (2, 1, 17, q) > 0\)
- \( p \) is prime with \( \left( \frac{-135}{p} \right) = -1 \).
Also we have

$$
\sum_{n>0} \frac{a(n)}{n^s} = \left( \frac{1}{1 + 5^{-s}} \right) \cdot \prod_r \frac{1}{(1 - r^{-s})^2} \\
\times \prod_t \frac{1}{(1 + t^{-s})^2} \cdot \prod_q \frac{1}{1 + q^{-s} + q^{-2s}} \\
\times \prod_w \frac{1}{1 - w^{-s} + w^{-2s}} \cdot \prod_p \frac{1}{1 - p^{-2s}}
$$

(8.7) \hspace{1cm} (8.8) \hspace{1cm} (8.9)

where $r$ is prime with $(1, 1, 34, r) > 0$

$t \neq 5$ is a prime with $(5, 5, 8, t) > 0$

$q$ is prime with $(4, 3, 9, q) > 0$

$w$ is a prime with $(2, 1, 17, w) > 0$

$p$ is prime with $\left( \frac{-135}{p} \right) = -1$. 
We can extract the Fourier coefficients of each product from $q E(q^9)E(q^{15}) + q^2 E(q^3)E(q^{45})$ by using congruences.
We can extract the Fourier coefficients of each product from $qE(q^9)E(q^{15}) + q^2E(q^3)E(q^{45})$ by using congruences. We have

$$[q^n]qE(q^9)E(q^{15}) = \begin{cases} a(n) & n \equiv 1 \pmod{3} \\ 0 & n \equiv 0, 2 \pmod{3}, \end{cases}$$

and

$$[q^n]q^2E(q^3)E(q^{45}) = \begin{cases} a(n) & n \equiv 2 \pmod{3} \\ 0 & n \equiv 0, 1 \pmod{3}. \end{cases}$$